Title	Forcing a morass with finite side conditions (Infinite Combinatorics and Forcing Theory)
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Citation	数理解析研究所講究録 = RIMS Kokyuroku (2017), 2042: 88-98
Issue Date	2017-07
URL	http://hdl.handle.net/2433/236944
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第2042巻 2017年 88-98

Forcing a morass with finite side conditions

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2017, March, 24

Abstract

We report a forcing poset that forces what we call a morass-type matrix. A condition of the poset is represented by a pair of a finite symmetric system of Aspero-Mota and a finite function from the finite symmetric system into the least uncountable cardinal. The finite function is a restriction of a rank function associated with a type of suitable countable symmetric system that contains the finite symmetric system. It is similar to forcing a club subset of the least uncountable cardinal by finite conditions that accompany finite $\in$-chains of elementary substructures. A difference between these two posets is whether cardinals can be preserved or not. Note that the forced matrix entails not just a club but a simplified morass of D. Velleman.

Notation

Let $(X, R, \cdots)$ be a structure, where $X \neq \emptyset$ is a set or a proper class, R is a binary relation, and so forth. Let Y be a non-empty, say, set with $Y \subseteq X$. We write $(Y, R, \cdots)$ or even Y for a substructure $(Y, R\cap(Y\times Y), \cdots)$ of $(X, R, \cdots)$. Let κ be a regular cardinal. Let $H_\kappa = \{x \mid$ the transitive closure of x is of a size $< \kappa\}$. We say N is a countable elementary substructure of $(H_\kappa, \in)$, if $(X, \in)$ is a countable elementary substructure of $(H_\kappa, \in)$. We use N, M, X, Y, Z and so forth for countable elementary substructures of $(H_{\omega_2}, \in)$. We use $\mathcal{N}$, $\mathcal{M}$, $\mathcal{A}$ and so forth for sets of countable elementary substructures of $(H_{\omega_2}, \in)$. When we write $X =_{\omega_1} Y$, this abbreviates $X \cap \omega_1 = Y \cap \omega_1$. When we write $X \geq_{\omega_1} Y$, this means $X \cap \omega_1 \geq Y \cap \omega_1$. Similarly for $X >_{\omega_1} Y$.

Introduction

We would like to explicate an idea behind our main forcing poset P by a prototype forcing poset Q. We first state well-known facts to avoid confusion.

Proposition. Let X, Y, and Z be countable elementary substructures of $(H_{\omega_2}, \in)$.

(1) If $x \in X$ and x is a countable set, then $x \subset X$.

(2) $X \cap \omega_1$ is a countable ordinal. Namely $N \cap \omega_1 < \omega_1$.

(3) If $Y \in X$, then $Y \subset X$.

(4) If $Z \in Y \in X$, then $Z \in X$. (transitive)

(5) $X \notin X$. (irreflexive)

Proof. (1): We may assume that x is non-empty. Since

$$(H_{\omega_2}, \in) \models \text{“}\exists\, e : \omega \longrightarrow x, e \text{ is onto”},$$

we have an enumeration $e : \omega \longrightarrow x$ with $e \in X$. Then

$$x = \{e(n) \mid n < \omega\} \subset X.$$

(2): We show that $X \cap \omega_1$ is transitive. Let $\alpha < \beta \in X \cap \omega_1$. We want $\alpha \in X \cap \omega_1$. Since $\beta \in X$ and β is countable, we have $\beta \subset X$ by (1). Hence $\alpha \in X \cap \omega_1$.

(3): Since $Y \in X$ and Y is countable, we have $Y \subset X$ by (1).

(4): Let $Z \in Y \in X$. Then $Z \in Y \subset X$ by (3). Hence $Z \in X$.

(5): We assume the axiom of regularity.

□

Let $\mathcal{N}$ be a non-empty set of countable elementary substructures of $(H_{\omega_2}, \in)$. We know that $(\mathcal{N}, \in)$ is a poset in the strong sense (irreflexive and transitive). We consider objects that generalize the countable ordinals. We say $\mathcal{N}$ is a continuous $\in$-chain, if

- ($\in$-chain, or, linear) If $Z, W \in \mathcal{N}$, then either $Z \in W$, $Z = W$, or $W \in Z$.
- (partitioned) If $Z \in \mathcal{N}$, then either $\mathcal{N} \cap Z = \emptyset$, $\exists Z_1\ \mathcal{N} \cap Z = \{Z_1\} \cup (\mathcal{N} \cap Z_1)$, or $\bigcup(\mathcal{N} \cap Z) = Z$.

Let $\mathcal{N}$ be a continuous $\in$-chain. Then the structure $(\mathcal{N}, \in)$ is a well-ordered one. Hence it makes sense to calculate the order types o.t.$(\mathcal{N}, \in)$ and o.t.$(\mathcal{N} \cap Z, \in)$ for each $Z \in \mathcal{N}$. We have o.t.$(\mathcal{N}, \in) \leq \omega_1$. If $\mathcal{N}$ is of a size finite, then it is clear that there are no differences between two concepts $\in$-chain (i.e, linear) and continuous $\in$-chain.

Let us next provide a prototype forcing poset Q that forces a club subset of ω_1. This Q is a variant to forcing a club subset of ω_1 by finite conditions due to J. E. Baumgartner.

Definition. Let $p = (\mathcal{N}^p, f^p) \in Q$, if

(ob) $\mathcal{N}^p$ is a finite $\in$-chain of countable elementary substructures of $(H_{\omega_2}, \in)$ and $f^p : \mathcal{N}^p \longrightarrow \omega_1$.

(wit) There exists a continuous $\in$-chain $\mathcal{M}$ of countable elementary substructures of $(H_{\omega_2}, \in)$ such that $\mathcal{M}$ is of a size countable, $\bigcup \mathcal{M} \in \mathcal{M}$ (a top element), $\mathcal{N}^p \subseteq \mathcal{M}$, and for each $Z \in \mathcal{N}^p$, $f^p(Z) =$ o.t.$(\mathcal{M} \cap Z, \in)$.

For $p, q \in Q$, let $q \leq p$ in Q, if $\mathcal{N}^q \supseteq \mathcal{N}^p$ and for each $Z \in \mathcal{N}^p$, $f^q(Z) = f^p(Z)$.

Theorem. (1) Let $p \in Q$, N^* be a countable elementary substructure of $(H_\theta, \in)$, θ is any sufficiently large regular cardinal, and $p, Q \in N^*$. Then

$$q = (\mathcal{N}^p \cup \{N^* \cap H_{\omega_2}\}, f^p \cup \{(N^* \cap H_{\omega_2}, N^* \cap \omega_1)\})$$

is (Q, N^*)-generic. Hence Q is proper.

(2) Let G be Q-generic over the ground model V. Let

$$\dot{\mathcal{N}} = \bigcup\{\mathcal{N}^p \mid p \in G\},$$

$$\dot{f} = \bigcup\{f^p \mid p \in G\}.$$

Then $\dot{\mathcal{N}}$ is a continuous $\in$-chain of countable elementary substructures of $(H^V_{\omega_2{}^V}, \in)$ such that

$$\text{o.t.}(\dot{\mathcal{N}}, \in) = \omega_1,$$

$$\bigcup \dot{\mathcal{N}} = H^V_{\omega_2^V},$$

and that for each $Z \in \dot{\mathcal{N}}$,

$$\dot{f}(Z) = \text{o.t.}(\dot{\mathcal{N}} \cap Z, \in).$$

In particular, ω_2^V gets collapsed.

□

We intend to force a simplified morass of [V] along this line of thought. Since we need to preserve ω_2, we resort to ideas from [A-M], [B-S], and [T]. This research was motivated by a talk by Borisa Kuzeljevic, Independence Results in Mathematics and Challenges in Iterated Forcing (UEA, Norwich, UK) 2015.

Preparation

We summarize two similar forcing posets P_{finite} and $P_{\text{countable}}$.

Definition. Let $p = \mathcal{N}^p \in P_{\text{finite}}$, if

(ob) $\mathcal{N}^p$ consists of countable elementary substructures of $(H_{\omega_2}, \in)$ and $\mathcal{N}^p$ is of a size finite.

(iso) For any $N, M \in \mathcal{N}^p$, if $N =_{\omega_1} M$, then there exists an (necessarily unique) isomorphism $\phi : (N, \in, \mathcal{N}^p \cap N) \longrightarrow (M, \in, \mathcal{N}^p \cap M)$ such that ϕ is the identity on the intersection $N \cap M$.

(up) If $N_3, N_2 \in \mathcal{N}^p$ with $N_3 <_{\omega_1} N_2$, then there exists $N_1 \in \mathcal{N}^p$ such that $N_3 \in N_1$ and $N_1 =_{\omega_1} N_2$.

For $p, q \in P_{\text{finite}}$, let $q \le p$ in P_{finite}, if $q \supseteq p$.

This notion of forcing due to, say, Aspero-Mota forces somewhat less than a morass that we call a matrix.

Theorem. ([AM]) (1) P_{finite} is proper and (CH) has the ω_2-cc.

(2) Let G be P_{finite}-generic over the ground model V and in $V[G]$, let

$$\dot{\mathcal{N}} = \bigcup G.$$

Then $\dot{\mathcal{N}}$ satisfies the following. And simply say that $\dot{\mathcal{N}}$ is a matrix.

(ob) $\dot{\mathcal{N}}$ consists of countable elementary substructures of $(H^V_{\omega_2}, \in)$.

(iso) For any $N, M \in \dot{\mathcal{N}}$, if $N =_{\omega_1} M$, then there exists an (necessarily unique) isomorphism $\phi : (N, \in, \dot{\mathcal{N}} \cap N) \longrightarrow (M, \in, \dot{\mathcal{N}} \cap M)$ such that ϕ is the identity on the intersection $N \cap M$.

(up) If $N_3, N_2 \in \dot{\mathcal{N}}$ with $N_3 <_{\omega_1} N_2$, then there exists $N_1 \in \dot{\mathcal{N}}$ such that $N_3 \in N_1$ and $N_1 =_{\omega_1} N_2$.

(stat) $\dot{\mathcal{N}}$ is stationary in $[H^V_{\omega_2}]^\omega$ and so $\in$-directed.

□

There is a way to get a quagmire of [K] by further forcing a club subset of the stationary set $\{N \cap \omega_1 \mid N \in \dot{\mathcal{N}}\}$ of ω_1 ([M1]).

The following has its roots in [BS].

Definition. Let $p = \mathcal{N}^p \in P_{\text{countable}}$, if

(ob) $\mathcal{N}^p$ consists of countable elementary substructures of $(H_{\omega_2}, \in)$ such that $\mathcal{N}^p$ is of a size countable and $N^p = \bigcup \mathcal{N}^p \in \mathcal{N}^p$ (a top element).

(iso) For any $N, M \in \mathcal{N}^p$, if $N =_{\omega_1} M$, then there exists an (necessarily unique) isomorphism $\phi : (N, \in, \mathcal{N}^p \cap N) \longrightarrow (M, \in, \mathcal{N}^p \cap M)$ such that ϕ is the identity on the intersection $N \cap M$.

(up) If $N_3, N_2 \in \mathcal{N}^p$ with $N_3 <_{\omega_1} N_2$, then there exists $N_1 \in \mathcal{N}^p$ such that $N_3 \in N_1$ and $N_1 =_{\omega_1} N_2$.

(par) $\mathcal{N}^p = \text{zero}(\mathcal{N}^p) \cup \text{suc}_1(\mathcal{N}^p) \cup \text{suc}_2(\mathcal{N}^p) \cup \lim(\mathcal{N}^p)$, where for $N \in \mathcal{N}^p$,

$$N \in \text{zero}(\mathcal{N}^p) \quad \text{iff} \quad N \cap \mathcal{N}^p = \emptyset,$$
$$N \in \text{suc}_1(\mathcal{N}^p) \quad \text{iff} \quad \exists N_1\ \mathcal{N}^p \cap N = \{N_1\} \cup (\mathcal{N}^p \cap N_1),$$
$$N \in \text{suc}_2(\mathcal{N}^p) \quad \text{iff} \quad \exists N_1 \exists N_2\ N_1 =_{\omega_1} N_2, (N_1, N_2) \models \Delta,$$
$$\mathcal{N}^p \cap N = \{N_1, N_2\} \cup (\mathcal{N}^p \cap N_1) \cup (\mathcal{N}^p \cap N_2),$$

where $(N_1, N_2) \models \Delta$ abbreviates that for $h \in N_1 \cap N_2 \cap \omega_2$, $t_1 \in (N_1 \cap \omega_2) \setminus N_2 \neq \emptyset$, $t_2 \in (N_2 \cap \omega_2) \setminus N_1 \neq \emptyset$, we have

$$h < t_1 < t_2 < \omega_2.$$

$$N \in \lim(\mathcal{N}^p) \quad \text{iff} \quad N = \bigcup(\mathcal{N}^p \cap N).$$

For $p, q \in P_{\text{countable}}$, let $q \leq p$ in $P_{\text{countable}}$, if $N^p \in \mathcal{N}^q$ and

$$\mathcal{N}^q \cap N^p = \mathcal{N}^p \cap N^p.$$

Since $\mathcal{N}^p = \{N^p\} \cup (\mathcal{N}^p \cap N^p)$ holds, $q \leq p$ in $P_{\text{countable}}$ iff $\mathcal{N}^q \supseteq \mathcal{N}^p$ and $\mathcal{N}^q \cap N^p = \mathcal{N}^p \cap N^p$.

Theorem. ([M2]) (1) $P_{\text{countable}}$ is proper, σ-Baire, and (CH) has the ω_2-cc.

(2) Let G be $P_{\text{countable}}$-generic over the ground model V and in $V[G]$, let

$$\dot{\mathcal{N}} = \bigcup G.$$

Then $\dot{\mathcal{N}}$ satisfies the following. And simply say that $\dot{\mathcal{N}}$ is a morass-type matrix.

(ob) $\dot{\mathcal{N}}$ consists of countable elementary substructures of $(H^V_{\omega_2}, \in)$.

(iso) For any $N, M \in \dot{\mathcal{N}}$, if $N =_{\omega_1} M$, then there exists an (necessarily unique) isomorphism $\phi : (N, \in, \dot{\mathcal{N}} \cap N) \longrightarrow (M, \in, \dot{\mathcal{N}} \cap M)$ such that ϕ is the identity on the intersection $N \cap M$.

(up) If $N_3, N_2 \in \dot{\mathcal{N}}$ with $N_3 <_{\omega_1} N_2$, then there exists $N_1 \in \dot{\mathcal{N}}$ such that $N_3 \in N_1$ and $N_1 =_{\omega_1} N_2$.

(par) $\dot{\mathcal{N}} = \text{zero}(\dot{\mathcal{N}}) \cup \text{suc}_1(\dot{\mathcal{N}}) \cup \text{suc}_2(\dot{\mathcal{N}}) \cup \lim(\dot{\mathcal{N}})$, where for $N \in \dot{\mathcal{N}}$,

$$N \in \text{zero}(\dot{\mathcal{N}}) \quad \text{iff} \quad N \cap \dot{\mathcal{N}} = \emptyset,$$

$$N \in \text{suc}_1(\dot{\mathcal{N}}) \quad \text{iff} \quad \exists N_1 \; \dot{\mathcal{N}} \cap N = \{N_1\} \cup (\dot{\mathcal{N}} \cap N_1),$$

$$N \in \text{suc}_2(\dot{\mathcal{N}}) \quad \text{iff} \quad \exists N_1 \exists N_2 \; N_1 =_{\omega_1} N_2, (N_1, N_2) \models \Delta,$$
$$\dot{\mathcal{N}} \cap N = \{N_1, N_2\} \cup (\dot{\mathcal{N}} \cap N_1) \cup (\dot{\mathcal{N}} \cap N_2),$$

$$N \in \lim(\dot{\mathcal{N}}) \quad \text{iff} \quad N = \bigcup(\dot{\mathcal{N}} \cap N).$$

(stat) $\dot{\mathcal{N}}$ is stationary in $[H^V_{\omega_2}]^\omega$ and so $\in$-directed.

□

There is a construction to get a simplified $(\omega_1, 1)$-morass out of this morass-type matrix $\dot{\mathcal{N}}$. Let us modify the assumption in section 6 of [M2] from LD(2) + Δ to LD(≤ 2) + Δ.

Theorem. ([M2]) Any morass-type matrix entails a simplified $(\omega_1, 1)$-morass.

□

Main Forcing

Here is the main forcing poset P that adds a morass-type matrix by finite side conditions. We know that any morass-type matrix entails a simplified $(\omega_1, 1)$-morass. For a condition $p \in P$, its main body is a function f^p. The domain $\mathcal{N}^p$ of f^p serves as a non-linear finite side condition.

Definition. Let $p = (\mathcal{N}^p, f^p) \in P$, if

(ob) $\mathcal{N}^p \in P_{\text{finite}}$ and $f^p : \mathcal{N}^p \longrightarrow \omega_1$.

(wit) There exists $\mathcal{M} \in P_{\text{countable}}$ such that $\mathcal{N}^p \subseteq \mathcal{M}$ and for all $N \in \mathcal{N}^p$,

$$f^p(N) = \rho^{\mathcal{M}}(N).$$

We refer to this situation (wit) as $p \in P$ witnessed by $\mathcal{M}$. Here, $\rho^{\mathcal{M}}$ is the rank function of the well-founded structure $(\mathcal{M}, \in)$. Since $\mathcal{M} \in P_{\text{countable}}$, we know that for all $M \in \mathcal{M}$,

$$\rho^{\mathcal{M}}(M) = \text{o.t.}\big(\{N \cap \omega_1 \mid N \in \mathcal{M} \cap M\}, <\big).$$

For $p, q \in P$, let $q \leq p$ in P, if $\mathcal{N}^q \leq \mathcal{N}^p$ in P_{finite} and for each $N \in \mathcal{N}^p$, $f^q(N) = f^p(N)$.

Hence, $q \leq p$ in P iff $f^q \supseteq f^p$. Note that there may exist many $\mathcal{M}$s in $P_{\text{countable}}$ for p and none of them are retained as parts of p. Hence, if we fix any choice $\mathcal{M}^p$ of $\mathcal{M}$ for p and any choice $\mathcal{M}^q$ of $\mathcal{M}$ for q, we do not expect to have $\mathcal{M}^q \leq \mathcal{M}^p$ in $P_{\text{countable}}$.

We next summarize on copying and pasting elements of P_{finite} and $P_{\text{countable}}$.

Lemma. (Copying and Pasting) Let X_1 and X_2 be two isomorphic countable elementary substructures of $(H_{\omega_2}, \in)$ such that the isomorphism $\phi_{X_1X_2} : (X_1, \in) \longrightarrow (X_2, \in)$ is the identity on the intersection $X_1 \cap X_2$.

(1) Let $\mathcal{N} \in X_1 \cap P_{\text{finite}}$ and let

$$\mathcal{N}' = \phi_{X_1X_2}[\mathcal{N}] = \{\phi_{X_1X_2}(Z) \mid Z \in \mathcal{N}\} = \{\phi_{X_1X_2}[Z] \mid Z \in \mathcal{N}\}.$$

Then $\mathcal{N}'$, $\mathcal{N} \cup \mathcal{N}'$, $\mathcal{N} \cup \{X_1\}$, $\mathcal{N}' \cup \{X_2\}$, and $\mathcal{N} \cup \mathcal{N}' \cup \{X_1, X_2\}$ are all in P_{finite}.

(2) Let $\mathcal{M} \in X_1 \cap P_{\text{countable}}$ and let

$$\mathcal{M}' = \phi_{X_1X_2}[\mathcal{M}] = \{\phi_{X_1X_2}(Z) \mid Z \in \mathcal{M}\} = \{\phi_{X_1X_2}[Z] \mid Z \in \mathcal{M}\}.$$

Then $\mathcal{M}'$, $\mathcal{M} \cup \{X_1\}$, and $\mathcal{M}' \cup \{X_2\}$ are all in $P_{\text{countable}}$. Furthermore, if $(X_1, X_2) \models \Delta$ and X is a countable elementary substructure of $(H_{\omega_2}, \in)$ with $X_1, X_2 \in X$. Then $\mathcal{M} \cup \mathcal{M}' \cup \{X_1, X_2, X\} \in P_{\text{countable}}$.

We mention facts on forming conditions in P. We just outline the last Lemma (Dense 3).

Lemma. (Dense 1) Let $p \in P$ witnessed by $\mathcal{M}$ and $Y \in \mathcal{M}$ such that $\mathcal{N}^p \in Y$. Then there exists $q \in P$ witnessed by $\mathcal{M}$ again such that $q \leq p$ in P and $Y \in \mathcal{N}^q$.

□

Lemma. (Dense 2) Let $p \in P$ witnessed by $\mathcal{M}$, $Y \in \mathcal{M}$, and $X_0 \in \mathcal{N}^p$ such that $\mathcal{N}^p \cap X_0 \in Y \in X_0$. Then there exists $q \in P$ witnessed by $\mathcal{M}$ again such that $q \leq p$ in P and $Y \in \mathcal{N}^q$.

□

Lemma. (Dense 3) Let $p \in P$ witnessed by $\mathcal{M}^p$, $X_0 \in \mathcal{N}^p$, $X_0 \in \text{suc}_2(\mathcal{M}^p)$, $X_1 =_{\omega_1} X_2$, $(X_1, X_2) \models \Delta$, and $\mathcal{M}^p \cap X_0 = \{X_1, X_2\} \cup (\mathcal{M}^p \cap X_1) \cup (\mathcal{M}^p \cap X_2)$. Then there exists $q \in P$ witnessed by $\mathcal{M}^p$ again such that $q \leq p$ in P, $X_1, X_2 \in \mathcal{N}^q$, and

$$\{Z \in \mathcal{N}^q \mid Z \geq_{\omega_1} X_0\} = \{Z \in \mathcal{N}^p \mid Z \geq_{\omega_1} X_0\}.$$

Proof. Let $\rho^{\mathcal{M}^p}(X_0) = i + 1$ and so $\rho^{\mathcal{M}^p}(X_1) = \rho^{\mathcal{M}^p}(X_2) = i$. We have two cases.

Case 1. $\mathcal{N}^p \cap X_0 = \emptyset$: For each $X \in \mathcal{N}^p$ with $X =_{\omega_1} X_0$, let $\mathcal{M}^q_X = \phi_{X_0X}[\{X_1, X_2\}] \cup \{X\}$. Let

$$\mathcal{N}^q = \{Z \in \mathcal{N}^p \mid Z \geq_{\omega_1} X_0\} \cup \bigcup \{\mathcal{M}^q_X \cap X \mid X \in \mathcal{N}^p, X =_{\omega_1} X_0\},$$

$$f^q = \rho^{\mathcal{M}^p} \lceil \mathcal{N}^q.$$

Let $q = (\mathcal{N}^q, f^q)$. Then this q works.

Case 2. $\mathcal{N}^p \cap X_0 \neq \emptyset$: Let

$$k = \max\{f^p(W) \mid W \in \mathcal{N}^p \cap X_0\}.$$

Then $k \leq i$ holds. We have two subcases.

Subcase 1. $k < i$: Let

$$\mathcal{A} = \{W \in \mathcal{N}^p \cap X_1 \mid f^p(W) = k\},$$

$$\mathcal{B} = \{W \in \mathcal{N}^p \cap X_2 \mid f^p(W) = k\},$$

$$\mathcal{C} = \mathcal{A} \cup \phi^{-1}_{X_1 X_2}[\mathcal{B}],$$

$$\mathcal{D} = \mathcal{C} \cup \phi_{X_1 X_2}[\mathcal{C}].$$

Then $\phi_{X_1 X_2}[\mathcal{C}] = (\phi_{X_1 X_2}[\mathcal{A}]) \cup \mathcal{B}$ and so

$$\mathcal{D} = (\mathcal{A} \cup \phi_{X_1 X_2}[\mathcal{A}]) \cup (\mathcal{B} \cup \phi_{X_2 X_1}[\mathcal{B}]),$$

$$X_1 \cap \mathcal{D} = \mathcal{C},$$

$$X_2 \cap \mathcal{D} = \phi_{X_1 X_2}[\mathcal{C}].$$

Step 1. Let $\mathcal{N}^{q_0} = \{Z \in \mathcal{N}^p \mid Z \geq_{\omega_1} X_0\} \cup \bigcup\{\phi_{X_0 X}[\{X_1, X_2\} \cup \mathcal{D}] \mid X \in \mathcal{N}^p, X =_{\omega_1} X_0\}$. Then $\mathcal{N}^{q_0} \leq \{X_0, X_1, X_2\} \cup \mathcal{D}$ in P_{finite}, $\{Z \in \mathcal{N}^{q_0} \mid Z \geq_{\omega_1} X_0\} = \{Z \in \mathcal{N}^p \mid Z \geq_{\omega_1} X_0\}$, and $\mathcal{N}^{q_0} \subset \mathcal{M}^p$.

Step 2. Let us fix any $W_0 \in \mathcal{A} \cup \mathcal{B}$ and let $\mathcal{N}^q_{W_0} = (\mathcal{N}^p \cap W_0) \cup \{W_0\}$. Let

$$\mathcal{N}^q_{X_0} = \{X_0, X_1, X_2\} \cup \bigcup\{\phi_{W_0 W}[\mathcal{N}^q_{W_0} \cap W_0] \cup \{W\} \mid W \in \mathcal{D}\}.$$

Then $\mathcal{N}^q_{X_0} \leq (\mathcal{N}^p \cap X_0) \cup \{X_0\}$ in P_{finite}, and $\mathcal{N}^q_{X_0} \subset \mathcal{M}^p$.

Step 3. Let $\mathcal{N}^q = \{Z \in \mathcal{N}^p \mid Z \geq_{\omega_1} X_0\} \cup \bigcup\{\phi_{X_0 X}[\mathcal{N}^q_{X_0} \cap X_0] \mid X \in \mathcal{N}^p, X =_{\omega_1} X_0\}$. Then $\mathcal{N}^q \leq \mathcal{N}^{q_0}, \mathcal{N}^p$ in P_{finite} and $\mathcal{N}^q \subset \mathcal{M}^p$.

Hence, $q = (\mathcal{N}^q, \rho^{\mathcal{M}^p} \lceil \mathcal{N}^q) \in P$ witnessed by $\mathcal{M}^p$ again, $q \leq p$ in P, $X_1, X_2 \in \mathcal{N}^q$, and

$$\{Z \in \mathcal{N}^q \mid Z \geq_{\omega_1} X_0\} = \{Z \in \mathcal{N}^p \mid Z \geq_{\omega_1} X_0\}.$$

Subcase 2. $i = k$: Let us fix $W_0 \in \{X_1, X_2\} \cap \mathcal{N}^p$ with $f^p(W_0) = k = i$. Let $\mathcal{N}^q_{W_0} = (\mathcal{N}^p \cap W_0) \cup \{W_0\}$. Let $\mathcal{N}^q_{X_0} = \{X_0, X_1, X_2\} \cup \bigcup\{\phi_{W_0 W}[\mathcal{N}^q_{W_0} \cap W_0] \mid W = X_1, X_2\}$. Let $\mathcal{N}^q = \{Z \in \mathcal{N}^p \mid Z \geq_{\omega_1} X_0\} \cup \bigcup\{\phi_{X_0 X}[\mathcal{N}^q_{X_0} \cap X_0] \mid X \in \mathcal{N}^p, X =_{\omega_1} X_0\}$. Then $q = (\mathcal{N}^q, \rho^{\mathcal{M}^p} \lceil \mathcal{N}^q)$ works.

□

We prepare a construction in $P_{\text{countable}}$.

Lemma. (Replace) Let $\mathcal{M}^d \in P_{\text{countable}}$, $X_0 \in \mathcal{M}^d$, $\mathcal{M} \in P_{\text{countable}}$ with $X_0 = \bigcup \mathcal{M}$. Then there exists $\mathcal{M}^s \in P_{\text{countable}}$ such that

- $\mathcal{M}^s \leq \mathcal{M}$ in $P_{\text{countable}}$,
- $\{Z \in \mathcal{M}^s \mid Z \geq_{\omega_1} X_0\} = \{Z \in \mathcal{M}^d \mid Z \geq_{\omega_1} X_0\}$.
- If $\rho^{\mathcal{M}^d}(X_0) = \rho^{\mathcal{M}}(X_0)$, then for all $Z \in \mathcal{M}^s$ with $Z \geq_{\omega_1} X_0$,

$$\rho^{\mathcal{M}^s}(Z) = \rho^{\mathcal{M}^d}(Z).$$

Proof. We want to replace the part $(\mathcal{M}^d \cap X_0) \cup \{X_0\}$ of $\mathcal{M}^d$ with $\mathcal{M}$ to form a new $\mathcal{M}^s \in P_{\text{countable}}$ that satisfies $X_0 \in \mathcal{M}^s$, $(\mathcal{M}^s \cap X_0) \cup \{X_0\} = \mathcal{M}$, and $\{Z \in \mathcal{M}^s \mid Z \geq_{\omega_1} X_0\} = \{Z \in \mathcal{M}^d \mid Z \geq_{\omega_1} X_0\}$. For each $X \in \mathcal{M}^d$ with $X =_{\omega_1} X_0$, let

$$\mathcal{M}^s_X = \{\phi_{X_0 X}(Y) \mid Y \in \mathcal{M} \cap X_0\} \cup \{X\}.$$

Let

$$\mathcal{M}^s = \{Z \in \mathcal{M}^d \mid Z \geq_{\omega_1} X_0\} \cup \bigcup \{\mathcal{M}^d_X \cap X \mid X \in \mathcal{M}^d, X =_{\omega_1} X_0\}.$$

Then this $\mathcal{M}^s$ works. In particular, $\mathcal{M}^s \cap X = \mathcal{M}^s_X \cap X$ for all $X \in \mathcal{M}^d$ with $X =_{\omega_1} X_0$.

□

Lemma. (Generic) Let $p \in P$ and N^* be a countable elementary substructure of $(H_\theta, \in)$ with $P, p \in N^*$, where θ is a sufficiently large regular cardinal. Let

$$q = (\mathcal{N}^p \cup \{N^* \cap H_{\omega_2}\}, f^p \cup \{(N^* \cap H_{\omega_2}, N^* \cap \omega_1)\}).$$

Then $q \in P$ such that $q \leq p$ and q is (P, N^*)-generic.

Proof. Let $X_0 = N^* \cap H_{\omega_2}$. To see that $q \in P$, let $p \in P$ witnessed by $\mathcal{M}^p$. We may assume that $\mathcal{M}^p \in N^*$. Let $\langle M_n \mid n < \omega \rangle$ be a $(P_{\text{countable}}, N^*)$-generic sequence with $\mathcal{M}_0 = \mathcal{M}^p$. Let

$$\mathcal{M}_\omega = \left(\bigcup \{\mathcal{M}_n \mid n < \omega\}\right) \cup \{X_0\}.$$

Then $\mathcal{M}_\omega \in P_{\text{countable}}$ with $X_0 = \bigcup \mathcal{M}_\omega$. Since $\mathcal{M}_\omega \leq \mathcal{M}^p$ in $P_{\text{countable}}$, we have $\rho^{\mathcal{M}^p} \subset \rho^{\mathcal{M}_\omega}$. Hence for all $Y \in \mathcal{N}^q \cap X_0 = \mathcal{N}^p$, we have

$$f^q(Y) = f^p(Y) = \rho^{\mathcal{M}^p}(Y) = \rho^{\mathcal{M}_\omega}(Y).$$

Since $\rho^{\mathcal{M}_\omega}(X_0) = X_0 \cap \omega_1$, we have

$$f^q(X_0) = \rho^{\mathcal{M}_\omega}(X_0).$$

Hence $q \in P$ witnessed by $\mathcal{M}_\omega$.

Let $D \in N^*$ be open dense in P. Let $d \leq q$. We may assume that $d \in D$. Want to find $d' \in D \cap N^*$ and $s \in P$ such that $s \leq d, d'$ in P. Note first that for all $Y \in \mathcal{N}^d \cap N^* = \mathcal{N}^d \cap X_0$, we have $f^d(Y) < f^d(X_0) = N^* \cap \omega_1$. Hence $f^d \cap N^* = \{(Y, f^d(Y)) \mid Y \in \mathcal{N}^d \cap N^*\}$. Since $D, \mathcal{N}^d \cap N^*, f^d \cap N^* \in N^*$ and N^* is an elementary substructure of $(H_\theta, \in)$, there exists $d' \in D \cap N^*$ such that

- $\mathcal{N}^d \cap N^* \subset \mathcal{N}^{d'}$,
- $f^d(Y) = f^{d'}(Y)$ for all $Y \in \mathcal{N}^d \cap N^*$.

Let $d' \in P$ witnessed by $\mathcal{M}' \in P_{\text{countable}} \cap N^*$. Let $\langle \mathcal{M}'_n \mid n < \omega \rangle$ be $(P_{\text{countable}}, N^*)$-generic sequence with $\mathcal{M}'_0 = \mathcal{M}'$. Let

$$\mathcal{M}'_\omega = \left(\bigcup \{\mathcal{M}'_n \mid n < \omega\}\right) \cup \{X_0\}.$$

Then $\mathcal{M}'_\omega \in P_{\text{countable}}$ with $\bigcup \mathcal{M}'_\omega = X_0$. By Lemma (Replace), we have $\mathcal{M}^s \in P_{\text{countable}}$ such that

- $\mathcal{M}^s \leq \mathcal{M}'_\omega$ in $P_{\text{countable}}$.
- $\{Z \in \mathcal{M}^s \mid Z \geq_{\omega_1} X_0\} = \{Z \in \mathcal{M}^d \mid Z \geq_{\omega_1} X_0\}$.

  But
- $\rho^{\mathcal{M}'_\omega}(X_0) = N^* \cap \omega_1 = f^q(X_0) = f^d(X_0) = \rho^{\mathcal{M}^d}(X_0)$.

  Hence
- $\rho^{\mathcal{M}^s}(Z) = \rho^{\mathcal{M}^d}(Z)$ for all $Z \in \mathcal{M}^s$ with $Z \geq_{\omega_1} X_0$.

For each $X \in \mathcal{M}^s$ with $X_0 =_{\omega_1} X$, let us write

$$\mathcal{M}^s_X = (\mathcal{M}^s \cap X) \cup \{X\}.$$

Hence we have,

$$\mathcal{M}^s_{X_0} = \mathcal{M}'_\omega,$$

$$\mathcal{M}^s_X = \{\phi_{X_0X}(Y) \mid Y \in \mathcal{M}^s_{X_0} \cap X_0\} \cup \{X\},$$

where $\phi_{X_0X} : X_0 \longrightarrow X$ is the isomorphism. Let

$$\mathcal{N}^s = \{Z \in \mathcal{N}^d \mid Z \geq_{\omega_1} X_0\} \cup \{\phi_{X_0X}(Y) \mid Y \in \mathcal{N}^{d'}, X \in \mathcal{N}^d, X =_{\omega_1} X_0\}.$$

For each $W \in \mathcal{N}^s$, let

$$f^s(W) = \rho^{\mathcal{M}^s}(W).$$

Let $s = (\mathcal{N}^s, f^s)$.

Claim. $s \in P$ witnessed by $\mathcal{M}^s$ and $s \leq d, d'$ in P.

Proof. Since $\mathcal{N}^d \in P_{\text{finite}}$ and $\mathcal{N}^d \cap N^* \subseteq \mathcal{N}^{d'} \in N^*$, it is routine to have $\mathcal{N}^s \in P_{\text{finite}}$.

We next observe that $\mathcal{N}^s \subseteq \mathcal{M}^s$. We have two cases. Let us first assume $Z \in \mathcal{N}^s$ with $Z \geq_{\omega_1} X_0$. Then $Z \in \mathcal{N}^d \subseteq \mathcal{M}^d$. Since $Z \geq_{\omega_1} X_0$, we have $Z \in \mathcal{M}^s$. Let us next assume that $Z <_{\omega_1} X_0$. Then there is $X \in \mathcal{N}^d$ such that $X =_{\omega_1} X_0$ and

$$Z \in \{\phi_{X_0X}(Y) \mid Y \in \mathcal{N}^{d'}\}.$$

But

$$\mathcal{N}^{d'} \subset \mathcal{M}'_\omega \cap X_0,$$

$$\phi_{X_0X}[\mathcal{M}'_\omega \cap X_0] = \mathcal{M}^s \cap X = \mathcal{M}^s_X \cap X.$$

In particular, $Z \in \mathcal{M}^s$.

We show that $s \leq d'$ in P. But by definition, we have that $\mathcal{N}^{d'} \subseteq \mathcal{N}^s$. Let $Z \in \mathcal{N}^{d'}$. Want $f^{d'}(Z) = f^s(Z)$. But

$$f^{d'}(Z) = \rho^{\mathcal{M}'}(Z) = \rho^{\mathcal{M}'_\omega}(Z) = \rho^{\mathcal{M}^s_{X_0}}(Z) = \rho^{\mathcal{M}^s}(Z) = f^s(Z).$$

Hence $s \leq d'$ in P.

We show that $s \leq d$ in P. Let $Z \in \mathcal{N}^d$. We have two cases. Let us first assume that $Z \geq_{\omega_1} X_0$. Then $Z \in \mathcal{N}^s$ by the definition of $\mathcal{N}^s$. We also have

$$f^d(Z) = \rho^{\mathcal{M}^d}(Z) = \rho^{\mathcal{M}^s}(Z) = f^s(Z).$$

Let us next assume that $Z <_{\omega_1} X_0$. Then there is $X \in \mathcal{N}^d$ such that $X =_{\omega_1} X_0$ and $Z \in \mathcal{M}^s_X \cap \mathcal{N}^d \subseteq \mathcal{N}^s$. We also have

$$f^d(Z) = f^d\big(\phi^{-1}_{X_0X}(Z)\big) = \rho^{\mathcal{M}^s_{X_0}}\big(\phi^{-1}_{X_0X}(Z)\big) = \rho^{\mathcal{M}^s_X}(Z) = \rho^{\mathcal{M}^s}(Z) = f^s(Z).$$

Hence $s \leq d$ in P.

$\square$

$\square$

Lemma. (CH) P has the ω_2-cc.

Proof. Let $\langle p_i \mid i < \omega_2 \rangle$ be a sequence of elements of P. For each $i < \omega_2$, let $p_i = (\mathcal{N}^{p_i}, f^{p_i}) \in P$ witnessed by $\mathcal{M}^{p_i}$ and let $M_i = \bigcup \mathcal{M}^{p_i}$. By CH, we may assume that there exist $i < j < \omega_2$ such that

- $(M_i, M_j) \models \Delta$,
- $\phi : (M_i, \in, \mathcal{M}^{p_i} \cap M_i, \mathcal{N}^{p_i}) \longrightarrow (M_j, \in, \mathcal{M}^{p_j} \cap M_j, \mathcal{N}^{p_j})$ is an isomorphism such that ϕ is the identity on the intersection $M_i \cap M_j$.

  Hence
- $\rho^{\mathcal{M}^{p_i}}(W) = \rho^{\mathcal{M}^{p_j}}(\phi(W))$ for all $W \in \mathcal{M}^{p_i}$.

  Let us fix any countable elementary substructure M of $(H_{\omega_2}, \in)$ with $p_i, p_j \in M$. Let

$$\mathcal{M} = \{M\} \cup \mathcal{M}^{p_i} \cup \mathcal{M}^{p_j},$$

$$\mathcal{N}^p = \mathcal{N}^{p_i} \cup \mathcal{N}^{p_j}.$$

  Then $\mathcal{M} \in P_{\text{countable}}$ such that $\mathcal{M} \leq \mathcal{M}^{p_i}, \mathcal{M}^{p_j}$ in $P_{\text{countable}}$. Hence

$$\rho^{\mathcal{M}} \supset \rho^{\mathcal{M}^{p_i}}, \rho^{\mathcal{M}^{p_j}}.$$

Let $p = (\mathcal{N}^p, \rho^{\mathcal{M}} \lceil \mathcal{N}^p)$. Then $p \in P$ witnessed by $\mathcal{M}$. This p is a common extension of p_i and p_j in P.

□

Lemma. (CH) Let G be P-generic over V. In $V[G]$, let us define

$$\dot{\mathcal{N}} = \bigcup \{\mathcal{N}^p \mid p \in G\}.$$

Then $\dot{\mathcal{N}}$ is a matrix. By this we mean that

(ob) $\dot{\mathcal{N}}$ consists of countable elementary substructures of $(H^V_{\omega_2}, \in)$.

(iso) For any $N, M \in \dot{\mathcal{N}}$, if $N =_{\omega_1} M$, then there exists a (necessarily unique) isomorphism $\phi : (N, \in, \dot{\mathcal{N}} \cap N) \longrightarrow (M, \in, \dot{\mathcal{N}} \cap M)$ such that ϕ is the identity on the intersection $N \cap M$.

(up) If $N_3, N_2 \in \dot{\mathcal{N}}$ with $N_3 <_{\omega_1} N_2$, then there exists $N_1 \in \dot{\mathcal{N}}$ such that $N_3 \in N_1$ and $N_1 =_{\omega_1} N_2$.

(stat) $\dot{\mathcal{N}}$ is stationary in $[H^V_{\omega_2}]^\omega$ and so $\in$-directed.

□

Since $(\dot{\mathcal{N}}, \in)$ is well-founded, the rank fuction $\rho^{\dot{\mathcal{N}}}$ is well-defined.

Lemma. Let G be P-generic over V. In $V[G]$, let us define

$$\dot{f} = \bigcup \{f^p \mid p \in G\}.$$

Let $p \in P$, $Z \in \mathcal{N}^p$, and $f^p(Z) = i$. Then there exists $q \leq p$ in P such that

- If $i = 0$, then $q \Vdash_P$ "$Z \in \text{zero}(\dot{\mathcal{N}})$ and $\rho^{\dot{\mathcal{N}}}(Z) = i$".
- If i is successor, then $q \Vdash_P$ "$Z \in \text{suc}_1(\dot{\mathcal{N}}) \cup \text{suc}_2(\dot{\mathcal{N}})$ and $\rho^{\dot{\mathcal{N}}}(Z) = i$".
- If i is limit, then $q \Vdash_P$ "$Z \in \lim(\dot{\mathcal{N}})$ and $\rho^{\dot{\mathcal{N}}}(Z) = i$".

In particular, $\dot{\mathcal{N}}$ is a morass-type matrix and $\dot{f}$ coincide with the rank function $\rho^{\dot{\mathcal{N}}}$ of the well-founded structure $(\dot{\mathcal{N}}, \in)$.

Proof. By induction on $i < \omega_1$. Let $p \in P$, $Z \in \mathcal{N}^p$, and $f^p(Z) = i$.

Case. $i = 0$: We claim $p \Vdash_P$ "$Z \in \text{zero}(\dot{\mathcal{N}})$ and so $\rho^{\dot{\mathcal{N}}}(Z) = 0$".

Proof. Suppose not. Let $q \leq p$ in P and $W \in Z \cap \mathcal{N}^q$. Then

$$f^q(W) < f^q(Z) = f^p(Z) = 0.$$

This would be a contradiction.

□

Case. $i = i + 1$: We have two subcases.

Subcase 1. For all $q \leq p$ and for all $\mathcal{M}$ such that $q \in P$ witnessed by $\mathcal{M}$, we have $Z \notin \mathrm{suc}_2(\mathcal{M})$: Let $p \in P$ witnessed by $\mathcal{M}^p$. Since $\rho^{\mathcal{M}^p}(Z) = f^p(Z) = i + 1$, we must have $Z \in \mathrm{suc}_1(\mathcal{M}^p) \cup \mathrm{suc}_2(\mathcal{M}^p)$. Since we are in Subcase 1, $Z \notin \mathrm{suc}_2(\mathcal{M}^p)$. Hence $Z \in \mathrm{suc}_1(\mathcal{M}^p)$. Let $\mathcal{M}^p \cap Z = \{Z_1\} \cup (\mathcal{M}^p \cap Z_1)$ and so $\rho^{\mathcal{M}^p}(Z_1) = i$. By Lemma (Dense 2), there exists $p' \leq p$ in P such that $p' \in P$ witnessed by $\mathcal{M}^p$ again and $Z_1 \in \mathcal{N}^{p'}$. Note that $f^{p'}(Z_1) = i$. By induction, we may assume, by extending p', that $p' \Vdash_P$ "$\rho^{\dot{\mathcal{N}}}(Z_1) = i$".

We claim $p' \Vdash_P$ "$\dot{\mathcal{N}} \cap Z = \{Z_1\} \cup (\dot{\mathcal{N}} \cap Z_1)$ and so $Z \in \mathrm{suc}_1(\dot{\mathcal{N}})$". Hence $p' \Vdash_P$ "$\rho^{\dot{\mathcal{N}}}(Z) = i + 1$".

Proof. Let $p'' \leq p'$ in P, $p'' \in P$ witnessed by $\mathcal{M}^{p''}$, and $W \in \mathcal{N}^{p''} \cap Z$. Suffices to show that either $W = Z_1$ or $W \in Z_1$. Since $\rho^{\mathcal{M}^{p''}}(Z) = f^{p''}(Z) = f^p(Z) = i+1$, we must have $Z \in \mathrm{suc}_1(\mathcal{M}^{p''}) \cup \mathrm{suc}_2(\mathcal{M}^{p''})$. Since we are in Subcase 1, $Z \notin \mathrm{suc}_2(\mathcal{M}^{p''})$. Hence $Z \in \mathrm{suc}_1(\mathcal{M}^{p''})$. But $\rho^{\mathcal{M}^{p''}}(Z) = i + 1$, $Z_1 \in \mathcal{N}^{p''} \cap Z \subseteq \mathcal{M}^{p''} \cap Z$, and $\rho^{\mathcal{M}^{p''}}(Z_1) = f^{p''}(Z_1) = f^{p'}(Z_1) = i$. Hence

$$W \in \mathcal{N}^{p''} \cap Z \subseteq \mathcal{M}^{p''} \cap Z = \{Z_1\} \cup (\mathcal{M}^{p''} \cap Z_1).$$

In particular, $W = Z_1$ or $W \in Z_1$.

□

Subcase 2. There are $q \leq p$ in P and $\mathcal{M}^q$ such that $q \in P$ witnessed by $\mathcal{M}^q$ and $Z \in \mathrm{suc}_2(\mathcal{M}^q)$: Let $Z_1 =_{\omega_1} Z_2$, $(Z_1, Z_2) \models \Delta$, and $\mathcal{M}^q \cap Z = \{Z_1, Z_2\} \cup (\mathcal{M}^q \cap Z_1) \cup (\mathcal{M}^q \cap Z_2)$. Then by Lemma (Dense 3), there is $q' \leq q$ in P, $q' \in P$ witnessed by $\mathcal{M}^q$ again, $Z_1, Z_2, Z \in \mathcal{N}^{q'}$, and $f^{q'}(Z_1) = f^{q'}(Z_2) = i < i + 1 = f^{q'}(Z) = f^q(Z)$. By induction, we may assume, by extending q' twice, that $q' \Vdash_P$ "$i = \rho^{\dot{\mathcal{N}}}(Z_1) = \rho^{\dot{\mathcal{N}}}(Z_2)$".

We claim $q' \Vdash_P$ "$\dot{\mathcal{N}} \cap Z = \{Z_1, Z_2\} \cup (\dot{\mathcal{N}} \cap Z_1) \cup (\dot{\mathcal{N}} \cap Z_2)$ and so $Z \in \mathrm{suc}_2(\dot{\mathcal{N}})$". Hence $q' \Vdash_P$ "$\rho^{\dot{\mathcal{N}}}(Z) = i + 1$".

Proof. Let $q'' \leq q'$ in P, $q'' \in P$ witnessed by $\mathcal{M}^{q''}$, and $W \in \mathcal{N}^{q''} \cap Z$. Suffices to show that either $W = Z_1$, $W = Z_2$, $W \in Z_1$, or $W \in Z_2$. Since $Z_1, Z_2, Z \in \mathcal{N}^{q''} \subseteq \mathcal{M}^{q''}$, we have $Z_1, Z_2, Z \in \mathcal{M}^{q''}$. But

$$\rho^{\mathcal{M}^{q''}}(Z_1) = \rho^{\mathcal{M}^{q''}}(Z_2) = f^{q''}(Z_1) = f^{q''}(Z_2) = i,$$

$$\rho^{\mathcal{M}^{q''}}(Z) = f^{q''}(Z) = f^p(Z) = i + 1,$$

and $(Z_1, Z_2) \models \Delta$. Hence $Z \in \mathrm{suc}_2(\mathcal{M}^{q''})$ and

$$W \in \mathcal{N}^{q''} \cap Z \subseteq \mathcal{M}^{q''} \cap Z = \{Z_1, Z_2\} \cup (\mathcal{M}^{q''} \cap Z_1) \cup (\mathcal{M}^{q''} \cap Z_2).$$

In particular, $W = Z_1$, $W = Z_2$, $W \in Z_1$, or $W \in Z_2$.

□

Case. i is limit: We claim $p \Vdash_P$ "$Z = \bigcup(\dot{\mathcal{N}} \cap Z)$ and so $Z \in \lim(\dot{\mathcal{N}})$".

Proof. Let $q \leq p$ in P, $q \in P$ wintnessed by $\mathcal{M}^q$, and $e \in Z$. Want to find $r \leq q$ in P such that there is $Y \in \mathcal{N}^r \cap Z$ with $e \in Y$. Since $i = f^p(Z) = f^q(Z) = \rho^{\mathcal{M}^q}(Z)$ is limit, we must have $Z \in \lim(\mathcal{M}^q)$. Hence $Z = \bigcup(\mathcal{M}^q \cap Z)$ and so there is $Y \in \mathcal{M}^q \cap Z$ such that $e, \mathcal{N}^q \cap Z \in Y$. By Lemma (Dense 2), there is $r \leq q$ in P such that $Y \in \mathcal{N}^r \cap Z$.

□

We claim $p \Vdash_P$ "$\rho^{\dot{\mathcal{N}}}(Z) = i$".

Proof. Suppose not. We argue in two cases.

Case 1. There are $q \leq p$ and $j < i$ such that $q \Vdash_P$ "$\rho^{\dot{\mathcal{N}}}(Z) = j$": Let $q \in P$ witnessed by $\mathcal{M}^q$. Since $i = f^q(Z) = \rho^{\mathcal{M}^q}(Z)$ is limit, we must have $Z \in \lim(\mathcal{M}^q)$. Hence there are $Y \in \mathcal{M}^q \cap Z$ and k such that $i > \rho^{\mathcal{M}^q}(Y) = k \geq j$ and $Z \cap \mathcal{N}^q \in Y$. By Lemma (Dense 2), we have $q' \leq q$ in P such that $Y \in \mathcal{N}^{q'}$ and $q' \in P$ is witnessed by $\mathcal{M}^q$ again. By induction there is $q'' \leq q'$ such that $q'' \Vdash_P$ "$\rho^{\dot{\mathcal{N}}}(Y) = k$". Since $Y \in Z$, we have $q'' \Vdash_P$ "$\rho^{\dot{\mathcal{N}}}(Z) > k \geq j$". But $q'' \leq q' \leq q$ in P and $q \Vdash_P$ "$\rho^{\dot{\mathcal{N}}}(Z) = j$". This would be a contradiction.

Case 2. There are $q \leq p$ in P and j such that $j > i$ and $q \Vdash_P$ "$\rho^{\dot{\mathcal{N}}}(Z) = j$": Take $q' \leq q$, $W \in \mathcal{N}^{q'} \cap Z$, and k such that $q' \Vdash_P$ "$\rho^{\dot{\mathcal{N}}}(W) = k \geq i$". By induction, we may assume, by extending q', that $f^{q'}(W) = k$. But $k < f^{q'}(Z) = f^p(Z) = i$. This would be a contradiction.

□

□

Theorem. (CH) Let G be P-generic over V. Then there exists a simplified $(\omega_1, 1)$-morass that is entailed from the morass-type matrix $\dot{\mathcal{N}} = \bigcup\{\mathcal{N}^p \mid p \in G\}$ in $V[G]$.

References

[A-M] D. Aspero, M. Mota, Forcing consequences of PFA together with the continuum large, *Transactions of the American Mathematical Society*, vol.367 (2015), no.9, pp.6103–6129.

[B-S] J. Baumgartner, S. Shelah, Remarks on superatomic Boolean algebras, *Annals of Pure and Applied Logic*, 33 (1987) pp.109–129.

[K] A. Kanamori, Morasses in combinatorial set theory, Surveys in Set Theory, *London Mathematical Society Lecture Notes Series*, 87 (1983) pp.167–196.

[M1] T. Miyamoto, Forcing a quagmire via matrices of models, a note, 2013.

[M2] T. Miyamoto, Matrices of isomorphic models and morass-like structures, a note, 2013.
http://www.kurims.kyoto-u.ac.jp/ kyodo/kokyuroku/contents/pdf/1895-09.pdf

[T] S. Todorcevic, Directed sets and cofinal types, *Transactions of the American Mathematical Society*, vol.290 (1985), no.2, pp.711–723.

[V] D. Velleman, Simplified morasses, *Journal of Symbolic Logic*, vol.49 (1984) no.1, pp.257–271.

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